**Review Problems – Introduction to Sensitivity Analysis (Python) *[See page 6 for Excel]***

**Joe Wilck**

Introducing Reduced Costs and Shadow Prices.

Decision Variables have **reduced costs**.

In linear programming, reduced cost, or opportunity cost, is the amount by which an objective function coefficient would have to improve (so increase for maximization problem, decrease for minimization problem) before it would be possible for a corresponding variable to assume a positive value in the optimal solution.

Constraints have **shadow prices**.

A shadow price of a resource constraint in linear programming is usually defined as the maximum price that should be paid to obtain an additional unit of resource. You can think of this shadow price as the “breakeven cost” of 1 additional unit of that constraint’s resource (all else being equal).

**Theme Park:**

Maintenance at a major theme park in Central Florida is an ongoing process that occurs 24 hours per day. Because it is a long drive from most residential areas to the park, employees do not like to work shifts fewer than eight hours. These 8-hour shifts start every four hours throughout the day, but the number of maintenance workers needed at different times throughout the day varies. The following table summarizes the number of employees needed in each four-hour time period.

  Time Period Minimum # of Employees

12AM – 4AM 90

4AM – 8AM 215

8AM – 12PM 250

12PM – 4PM 165

4PM – 8PM 300

8PM – 12AM 125

The maintenance supervisor wants to determine the minimum number of employees to schedule for each shift in order to meet staffing requirements.

**Solution:**

Classification: Scheduling Problem

*Algebraic Formulation:*

Decision Variables:

S0000: number of maintenance workers whose shift starts at 0000 and ends at 0800

S0400: number of maintenance workers whose shift starts at 0400 and ends at 1200

S0800: number of maintenance workers whose shift starts at 0800 and ends at 1600

S1200: number of maintenance workers whose shift starts at 1200 and ends at 2000

S1600: number of maintenance workers whose shift starts at 1600 and ends at 0000

S2000: number of maintenance workers whose shift starts at 2000 and ends at 0400

Objective: Minimize Number of Workers

Minimize S0000 + S0400 + S0800 + S1200 + S1600 + S2000

*(Note that the coefficients of this objective are one for all variables.)*

Constraints:

S0000 + S2000 ≥ 90 *[Ensuring capacity for 0000 to 0400]*

S0000 + S0400 ≥ 215 *[Ensuring capacity for 0400 to 0800]*

S0400 + S0800 ≥ 250 *[Ensuring capacity for 0800 to 1200]*

S0800 + S1200 ≥ 165 *[Ensuring capacity for 1200 to 1600]*

S1200 + S1600 ≥ 300 *[Ensuring capacity for 1600 to 2000]*

S1600 + S2000 ≥ 125 *[Ensuring capacity for 2000 to 0000]*

*(Note the above constraints satisfy the maintenance worker requirements for each 4 hour period.)*

S0000, S0400, S0800, S1200, S1600, S2000 ≥ 0 *(Non-negativity constraints)*

**Python Output:**

Problem 1

status=Optimal

S0000 = 0.0

S0400 = 250.0

S0800 = 0.0

S1200 = 265.0

S1600 = 35.0

S2000 = 90.0

Objective = 640.0

S0000 = 0.0 Reduced Cost = 0.0

S0400 = 250.0 Reduced Cost = 0.0

S0800 = 0.0 Reduced Cost = 0.0

S1200 = 265.0 Reduced Cost = 0.0

S1600 = 35.0 Reduced Cost = 0.0

S2000 = 90.0 Reduced Cost = 0.0

*\*In a “normal” problem (i.e., not alt. optimal) you will not see a 0 for the variable value and a corresponding 0 for a reduced cost.*

*\*Note, for optimal solution, if the variable value is > 0; then its corresponding reduced cost = 0.*

Shadow prices are for the constraints

C6 0 in shadow price and slack means there is an alternative solution as in a normal problem this don’t happen.

name shadow price slack

0 \_C1 1.0 -0.0

1 \_C2 0.0 -35.0

2 \_C3 1.0 -0.0

3 \_C4 0.0 -100.0

4 \_C5 1.0 -0.0

5 \_C6 0.0 -0.0

*\*In a “normal” problem (i.e., not alt. optimal) you will not see a 0 for shadow price and a corresponding 0 for slack.*

*Sensitivity Analysis:*

Interpretation:

First, with respect to the Variables, S0400, S1200, S1600, and S2000 all have workers assigned to those decision variables. However, S0000 and S0800 do not (currently have values of 0). However, notice that S0000 and S0800 have reduced costs of 0. Thus, there are alternative optimal solutions where we could include a worker (or more) where some did start at midnight (S0000) or 8AM (S0800).

Second, with respect to the Constraints, all four-hour time periods are satisfied at their binding level except for 0400-0800 and 1200-1600 (which has 300 workers but only a need for 165). However, notice that for the 2000-0000 constraint it is binding but also has a shadow price of 0. This is an indication of alternative optimal solutions.

(Note, you don’t have to have an indication from both the Variables and Constraints tables to have Alternative Optimal Solutions, just from one, but this is an example where you have them from both.)

One of many alternative optimal solutions:



*Note, alternative optimal solutions (all add up to 640):*

You’ll notice as we add S2000, we reduce S0000, increase S0400, reduce S0800, increase S1200, and reduce S1600 by the equivalent amount.







**Wine Shipment:**

A winery has the following capacity to produce an exclusive dinner wine at either of its two vineyards at the indicated costs:

Vineyard A: 3500 bottles at $23 per bottle

Vineyard B: 3100 bottles at $25 per bottle

Four Italian restaurants around the country are interested in purchasing the wine. Each restaurant has limited demand for the wine and will pay the following prices:

Restaurant 1: 1800 bottles at $69 per bottle

Restaurant 2: 2300 bottles at $67 per bottle

Restaurant 3: 1250 bottles at $70 per bottle

Restaurant 4: 1750 bottles at $66 per bottle

The costs of shipping a bottle from the vineyards to the restaurant are as follows:

Restaurant 1 Restaurant 2 Restaurant 3 Restaurant 4

Vineyard A: $7 $8 $13 $11

Vineyard B: $12 $6 $8 $12

The winery needs to determine the production and shipping plan that maximizes profits.

A picture always helps……



*Drawing of Problem (notice variables are the arcs)*

Classification: Transportation Problem

*Algebraic Formulation:*

Decision Variables:

A1: number of wine bottles sent from Vineyard A to Restaurant 1

A2: number of wine bottles sent from Vineyard A to Restaurant 2

…

B4: number of wine bottles sent from Vineyard B to Restaurant 4

Objective: Maximize Profit

Maximize 39A1 + 36A2 + 34A3 + 32A4 + 32B1 + 36B2 + 37B3 + 29B4

*(Note, we get our profit values by subtracting costs from revenue. This is done step-by-step in Excel.)*

Constraints:

A1 + A2 + A3 + A4 ≤ 3500 [Vineyard A’s Supply Constraint]

B1 + B2 + B3 + B4 ≤ 3100 [Vineyard B’s Supply Constraint]

A1 + B1 ≤ 1800 [Restaurant 1’s Demand Constraint]

A2 + B2 ≤ 2300 [Restaurant 2’s Demand Constraint]

A3 + B3 ≤ 1250 [Restaurant 3’s Demand Constraint]

A4 + B4 ≤ 1750 [Restaurant 4’s Demand Constraint]

A1, A2, A3, A4, B1, B2, B3, B4 ≥ 0 [Non-negativity constraints]

**Python Output:**

Problem 2

status=Optimal

A1 = 1800.0

A2 = 450.0

A3 = 0.0

A4 = 1250.0

B1 = 0.0

B2 = 1850.0

B3 = 1250.0

B4 = 0.0

Objective = 239250.0

A1 = 1800.0 Reduced Cost = -0.0

A2 = 450.0 Reduced Cost = -0.0

A3 = 0.0 Reduced Cost = -3.0 profit will go down by 3 dollars

A4 = 1250.0 Reduced Cost = -0.0

B1 = 0.0 Reduced Cost = -7.0

B2 = 1850.0 Reduced Cost = -0.0

B3 = 1250.0 Reduced Cost = -0.0

B4 = 0.0 Reduced Cost = -3.0

name shadow price slack

0 \_C1 32.0 -0.0

1 \_C2 32.0 -0.0

2 \_C3 7.0 -0.0

3 \_C4 4.0 -0.0

4 \_C5 5.0 -0.0

5 \_C6 -0.0 500.0

*Sensitivity Analysis:*

First, notice from the Variable cells, that you are using A1, A2, A4, B2, and B3. The other variables have a Final Value of 0 bottles, and negative Reduced Costs. That means that they will reduce your profit if they were in the solution (e.g., sending 1 bottle from Vineyard A to Restaurant 3 would reduce your profit by $3).

Second, notice from the Constraints, that you are using all of your Supply (you have excess demand) since both Vineyard Supply constraints are binding. You’ll notice you are satisfying Restaurants 1-3’s demand (those constraints are binding), but not Restaurant 4’s demand (you have 500 spare demand). This is because (looking back at the Variable table), A4 and B4 do not give you as much revenue as the other variables. In other words, Restaurant 4 does not generate as much profit per bottle than the other Restaurants.

**Review Problems – Introduction to Sensitivity Analysis (Excel)**

**Joe Wilck**

**Excel:**

This review goes along with the Theme Park, Hot Dog, and Wine Shipment problems. Refer to the Excel file for the original problem statement, etc.

**Video:**

This review corresponds to the following videos:

Theme Park: <https://youtu.be/pQGbhU-4_Uo>

Hot Dog: <https://youtu.be/bZcUww80yGM>

Wine Shipment: <https://youtu.be/iu-2sCyElKs>

**Theme Park:**

Classification: Scheduling Problem

*Algebraic Formulation:*

Decision Variables:

S0000: number of maintenance workers whose shift starts at 0000 and ends at 0800

S0400: number of maintenance workers whose shift starts at 0400 and ends at 1200

S0800: number of maintenance workers whose shift starts at 0800 and ends at 1600

S1200: number of maintenance workers whose shift starts at 1200 and ends at 2000

S1600: number of maintenance workers whose shift starts at 1600 and ends at 0000

S2000: number of maintenance workers whose shift starts at 2000 and ends at 0400

Objective: Minimize Number of Workers

Minimize S0000 + S0400 + S0800 + S1200 + S1600 + S2000

*(Note that the coefficients of this objective are one for all variables.)*

Constraints:

S0000 + S2000 ≥ 90 *[Ensuring capacity for 0000 to 0400]*

S0000 + S0400 ≥ 215 *[Ensuring capacity for 0400 to 0800]*

S0400 + S0800 ≥ 250 *[Ensuring capacity for 0800 to 1200]*

S0800 + S1200 ≥ 165 *[Ensuring capacity for 1200 to 1600]*

S1200 + S1600 ≥ 300 *[Ensuring capacity for 1600 to 2000]*

S1600 + S2000 ≥ 125 *[Ensuring capacity for 2000 to 0000]*

*(Note the above constraints satisfy the maintenance worker requirements for each 4 hour period.)*

S0000, S0400, S0800, S1200, S1600, S2000 ≥ 0 *(Non-negativity constraints)*

**Excel solution, etc. see Excel file.**

*Sensitivity Analysis:*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Microsoft Excel 14.0 Sensitivity Report** | | |  |  |  |  |  |  |
| **Worksheet: [Theme Park Hot Dog Wine.xlsx]Theme Park** | | | | | | | |  |
| **Report Created:** | | |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Variable Cells | | |  |  |  |  |  |  |
|  |  |  | **Final** | **Reduced** | **Objective** | **Allowable** | **Allowable** |  |
|  | **Cell** | **Name** | **Value** | **Cost** | **Coefficient** | **Increase** | **Decrease** |  |
|  | $C$4 | S0000 | 90 | 0 | 1 | 0 | 1 |  |
|  | $D$4 | S0400 | 125 | 0 | 1 | 1 | 0 |  |
|  | $E$4 | S0800 | 125 | 0 | 1 | 0 | 1 |  |
|  | $F$4 | S1200 | 175 | 0 | 1 | 0 | 0 |  |
|  | $G$4 | S1600 | 125 | 0 | 1 | 0 | 0 |  |
|  | $H$4 | S2000 | 0 | 0 | 1 | 1E+30 | 0 |  |
|  |  |  |  |  |  |  |  |  |
| Constraints | | |  |  |  |  |  |  |
|  |  |  | **Final** | **Shadow** | **Constraint** | **Allowable** | **Allowable** |  |
|  | **Cell** | **Name** | **Value** | **Price** | **R.H. Side** | **Increase** | **Decrease** |  |
|  | $I$9 | Required 0000 - 0400 Objective Cell | 90 | 1 | 90 | 125 | 90 |  |
|  | $I$10 | Required 0400 - 0800 Objective Cell | 215 | 0 | 215 | 125 | 125 |  |
|  | $I$11 | Required 0800 - 1200 Objective Cell | 250 | 1 | 250 | 1E+30 | 125 |  |
|  | $I$12 | Required 1200 - 1600 Objective Cell | 300 | 0 | 165 | 135 | 1E+30 |  |
|  | $I$13 | Required 1600 - 2000 Objective Cell | 300 | 1 | 300 | 1E+30 | 135 |  |
|  | $I$14 | Required 2000 - 0000 Objective Cell | 125 | 0 | 125 | 135 | 125 |  |
|  |  |  |  |  |  |  |  |  |

Interpretation:

First, with respect to the Variables, S0000, S0400, S0800, S1200, and S1600 all have workers assigned to those decision variables. However, S2000 does not (currently has a value of 0). However, notice that S2000 has a Reduced Cost of 0. Thus, there are alternative optimal solutions where we could include a worker (or more) where some did start at 8:00PM.

Second, with respect to the Constraints, all four-hour time periods are satisfied at their binding level except for 1200-1600 (which has 300 workers but only a need for 165). However, notice that for the 0400-0800 and 2000-0000 constraints they are both binding but also have a reduced price of 0. This is an indication of alternative optimal solutions.

(Note, you don’t have to have an indication from both the Variables and Constraints tables to have Alternative Optimal Solutions, just from one, but this is an example where you have them from both.)

Note, alternative optimal solutions (all add up to 640):



You’ll notice as we add S2000, we reduce S0000, increase S0400, reduce S0800, increase S1200, and reduce S1600 by the equivalent amount.





We can do this up to S2000 = 90 (which reduces S0000 to zero).



**Hot Dog:**

Classification: Product Mix Problem

*Algebraic Formulation:*

Decision Variables:

Beef: pounds of beef in hot dog

Pork: pounds of pork in hot dog

Chicken: pounds of chicken in hot dog

Turkey: pounds of turkey in hot dog

Objective: Minimize Cost of Hot Dog

Minimize 0.76(Beef) + 0.82(Pork) + 0.64(Chicken) + 0.58(Turkey)

Constraints:

640(Beef) + 1055(Pork) + 780(Chicken) + 528(Turkey) ≤ 100 [Calorie constraint]

32.5(Beef) + 54(Pork) + 25.6(Chicken) + 6.4(Turkey) ≤ 6 [Fat constraint]

210(Beef) + 205(Pork) + 220(Chicken) + 172(Turkey) ≤ 27 [Cholesterol constraint]

Beef ≥ 0.25(Beef + Pork + Chicken + Turkey) [Beef must be ≥25% of total weight]

Pork ≥ 0.25(Beef + Pork + Chicken + Turkey) [Pork must be ≥25% of total weight]

Beef + Pork + Chicken + Turkey ≥ 2/16 [Hot Dog must be at least 2 ounces]

*(Note, for the last constraint, 1/16 of a pound is one ounce. The right-hand side includes the conversion.)*

Beef, Pork, Chicken, Turkey ≥ 0 *(Non-negativity constraints)*

**Excel solution, etc. see Excel file.**

*Sensitivity Analysis:*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Microsoft Excel 14.0 Sensitivity Report** | | | |  |  |  |  |
| **Worksheet: [Theme Park Hot Dog Wine.xlsx]hot dog** | | | | | | | |
| **Report Created:** | | | |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
| Variable Cells | | |  |  |  |  |  |
|  |  |  | **Final** | **Reduced** | **Objective** | **Allowable** | **Allowable** |
|  | **Cell** | **Name** | **Value** | **Cost** | **Coefficient** | **Increase** | **Decrease** |
|  | $C$4 | Beef | 0.03125 | 0 | 0.76 | 1E+30 | 0.18 |
|  | $D$4 | Pork | 0.03125 | 0 | 0.82 | 1E+30 | 0.24 |
|  | $E$4 | Chicken | 0 | 0.06 | 0.64 | 1E+30 | 0.06 |
|  | $F$4 | Turkey | 0.0625 | 0 | 0.58 | 0.06 | 1.37 |
|  |  |  |  |  |  |  |  |
| Constraints | | |  |  |  |  |  |
|  |  |  | **Final** | **Shadow** | **Constraint** | **Allowable** | **Allowable** |
|  | **Cell** | **Name** | **Value** | **Price** | **R.H. Side** | **Increase** | **Decrease** |
|  | $G$12 | Beef USED | 0.03125 | 0.18 | 0 | 0.0625 | 0.03125 |
|  | $G$13 | Pork USED | 0.03125 | 0.24 | 0 | 0.026624763 | 0.03125 |
|  | $G$14 | Total Weight (lbs) USED | 0.125 | 0.685 | 0 | 0.01729249 | 0.125 |
|  | $G$9 | Calories/lb USED | 85.96875 | 0 | 100 | 1E+30 | 14.03125 |
|  | $G$10 | Fat (g/lb) USED | 3.103125 | 0 | 6 | 1E+30 | 2.896875 |
|  | $G$11 | Cholesterol (g/lb) USED | 23.71875 | 0 | 27 | 1E+30 | 3.28125 |

Interpretation:

First, with respect to the Variables, we use Beef, Pork, and Turkey in the solution, but do not use Chicken. Chicken has a reduced cost of 0.06 per pound. Basically, it is 6 cents per pound more expensive than turkey; thus, turkey is preferred. Notice we are using the minimum amount of Beef and Pork to satisfy the 25% requirement and we are making exactly a 2 ounce hot dog.

Second, with respect to the Constraints, you’ll notice a “0” for Current R.H. Side for the 25% constraints for Beef and Pork and for Total Weight [see orange oval above]. That is because we entered the Right-Hand Side as a formula for those three cells. If we had hard-coded in the minimums, then we would have values other than 0 in those places. Those three constraints are binding (if you look at your Excel Solution you’ll notice the Gray Cell (Left-Hand side total) matches the Right-Hand side total). You’ll notice we are not binding on our nutritional constraints (Calories, Fat, and Cholesterol); thus, they have a Shadow Price of 0.

**Wine Shipment:**

Classification: Transportation Problem

Drawing of Problem (notice variables are the arcs):



*Algebraic Formulation:*

Decision Variables:

A1: number of wine bottles sent from Vineyard A to Restaurant 1

A2: number of wine bottles sent from Vineyard A to Restaurant 2

…

B4: number of wine bottles sent from Vineyard B to Restaurant 4

Objective: Maximize Profit

Maximize 39A1 + 36A2 + 34A3 + 32A4 + 32B1 + 36B2 + 37B3 + 29B4

*(Note, we get our profit values by subtracting costs from revenue. This is done step-by-step in Excel.)*

Constraints:

A1 + A2 + A3 + A4 ≤ 3500 [Vineyard A’s Supply Constraint]

B1 + B2 + B3 + B4 ≤ 3100 [Vineyard B’s Supply Constraint]

A1 + B1 ≤ 1800 [Restaurant 1’s Demand Constraint]

A2 + B2 ≤ 2300 [Restaurant 2’s Demand Constraint]

A3 + B3 ≤ 1250 [Restaurant 3’s Demand Constraint]

A4 + B4 ≤ 1750 [Restaurant 4’s Demand Constraint]

A1, A2, A3, A4, B1, B2, B3, B4 ≥ 0 [Non-negativity constraints]

**Excel solution, etc. see Excel file.**

*Sensitivity Analysis:*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Microsoft Excel 14.0 Sensitivity Report** | | |  |  |  |  |  |
| **Worksheet: [Theme Park Hot Dog Wine.xlsx]Wine Shipments** | | | | | | | |
| **Report Created:** | | |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
| Variable Cells | | |  |  |  |  |  |
|  |  |  | **Final** | **Reduced** | **Objective** | **Allowable** | **Allowable** |
|  | **Cell** | **Name** | **Value** | **Cost** | **Coefficient** | **Increase** | **Decrease** |
|  | $C$4 | A1 | 1800 | 0 | 39 | 1E+30 | 7 |
|  | $D$4 | A2 | 450 | 0 | 36 | 3 | 3 |
|  | $E$4 | A3 | 0 | -3 | 34 | 3 | 1E+30 |
|  | $F$4 | A4 | 1250 | 0 | 32 | 4 | 3 |
|  | $G$4 | B1 | 0 | -7 | 32 | 7 | 1E+30 |
|  | $H$4 | B2 | 1850 | 0 | 36 | 3 | 3 |
|  | $I$4 | B3 | 1250 | 0 | 37 | 1E+30 | 3 |
|  | $J$4 | B4 | 0 | -3 | 29 | 3 | 1E+30 |
|  |  |  |  |  |  |  |  |
| Constraints | | |  |  |  |  |  |
|  |  |  | **Final** | **Shadow** | **Constraint** | **Allowable** | **Allowable** |
|  | **Cell** | **Name** | **Value** | **Price** | **R.H. Side** | **Increase** | **Decrease** |
|  | $K$12 | Vineyard A Supply Objective Cell | 3500 | 32 | 3500 | 500 | 1250 |
|  | $K$13 | Vineyard B Supply Objective Cell | 3100 | 32 | 3100 | 450 | 1250 |
|  | $K$14 | Restaurant 1 Demand Objective Cell | 1800 | 7 | 1800 | 1250 | 500 |
|  | $K$15 | Restaurant 2 Demand Objective Cell | 2300 | 4 | 2300 | 1250 | 450 |
|  | $K$16 | Restaurant 3 Demand Objective Cell | 1250 | 5 | 1250 | 1250 | 450 |
|  | $K$17 | Restaurant 4 Demand Objective Cell | 1250 | 0 | 1750 | 1E+30 | 500 |

First, notice from the Variable cells, that you are using A1, A2, A4, B2, and B3. The other variables have a Final Value of 0 bottles, and negative Reduced Costs. That means that they will reduce your profit if they were in the solution (e.g., sending 1 bottle from Vineyard A to Restaurant 3 would reduce your profit by $3).

Second, notice from the Constraints, that you are using all of your Supply (you have excess demand) since both Vineyard Supply constraints are binding. You’ll notice you are satisfying Restaurants 1-3’s demand (those constraints are binding), but not Restaurant 4’s demand (you have 500 spare demand). This is because (looking back at the Variable table), A4 and B4 do not give you as much revenue as the other variables. In other words, Restaurant 4 does not generate as much profit per bottle than the other Restaurants.

*Excerpt from* ***Linear Programming and Extensions*** *(George B. Dantzig, 1963; pages 321-322):*

- The Marriage Game (Assigning men to women based on preference).

- Story (copied from page 322):

In 1955, at the summer meeting of the Operations Research Society in Los Angeles, I (Dantzig) was interviewed by the press. The reporter turned out to be the brother of my small daughter's piano teacher, and so we became quite friendly. I explained to him that linear programming models originated in the Air Force, and I described their growing application to industrial problems. It became obvious that this veteran Hollywood reporter was having a hard time seeing how to make the material into an exciting news story. In desperation I suggested, "How about something with sex appeal?" "Now you're talking," he said. "Well," I continued, "an interesting by-product of our work with linear programming models is a mathematical proof that of all the possible forms of marriage (monogamy, bigamy, polygamy, etc.), monogamy is the best." "You say monogamy is the best of all possible relations?" he queried. "Yes," I replied. "Man," he said, shaking his head in the negative, "you've been working with the wrong kind of models."